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RANDOM VIBRATION OF A NONLINEARLY DEFORMED BEAM BY A NEW STOCHASTIC LINEARIZATION TECHNIQUE

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Abstract—A new stochastic linearization technique is employed to investigate the large amplitude random vibrations of a simply supported or a clamped beam on elastic foundation under a stochastic loading which is space-wise either (a) white noise or (b) uniformly distributed load and time-wise white noise. The new version of the stochastic linearization method is based on the requirement that the mean square deviation of the strain energy of the nonlinearly deformed beam, and the strain energy of the equivalent beam in a linear state, should be minimal. As a result, the modal mean square displacements are expressed as solutions of a set of nonlinear algebraic equations. Results obtained by the conventional equivalent linearization method and by the new technique are compared with the numerical results obtained from integration of the exact probability density function (when the exact solution is available) or with the result of the Monte Carlo simulations (when the exact solution is unavailable). It is shown that the new stochastic linearization technique yields a much more accurate estimate of the mean square displacement than the classical linearization method, which has attracted the past interest of about 400 investigators in a variety of nonlinear random vibration problems.

1. INTRODUCTION

The random vibrations of beams in linear and nonlinear settings have been investigated by several authors. Linear random vibrations have been investigated by Eringen (1957), Bogdanoff and Goldberg (1960), Crandall and Yildiz (1962), Elishakoff and Livshits (1984) and Elishakoff (1987), by employing the normal mode method. Eringen (1957), Elishakoff (1987) and Elishakoff and Livshits (1984) were able to sum up the infinite series of modal contributions and derive closed-form solutions for simply supported beams subjected to loading which is both time-wise and space-wise white noise. For this particular excitation, Herbert (1964, 1965) succeeded in obtaining an exact, although not a closed-form solution, for probability density function of modal displacements. For the general case of excitation, when dealing with the nonlinear stochastic boundary value problems, most investigators have employed approximate techniques: either the classical perturbation method or the stochastic linearization technique. The latter method has attracted numerous investigators. Indeed the only monograph on this subject is that by Roberts and Spanos (1990) which lists approximately 250 papers utilizing the stochastic linearization technique. The review by Sinitsin (1974) lists about 120 studies predominantly performed in Russia. The review paper by Socha and Soong (1991) lists numerous publications written in both the West and East. Thus presently there are approximately 400 studies based on the classical stochastic linearization technique. Unfortunately, these methods have severe limitations: the perturbation method usually only leads to results of acceptable accuracy for the case of small nonlinearity; stochastic linearization technique can yield an error as large as 20% in the estimate of the mean square response for some cases of nonlinearity and excitation. It should be borne in mind that even an exact solution by the Fokker–Planck equation method may involve a large amount of multiple integrations to evaluate the mean square responses if many modes need to be included. The latter difficulty is of purely numerical nature whereas the approximate methods have their inherent difficulties. Furthermore, the fact that the (numerically cumbersome) exact solution is available for extremely specific cases of excitations rules out its general application. The above disadvantages of the approximate methods and the absence of the exact solution for the general loading case encouraged the present authors to seek an alternative approximate technique. An improved stochastic linearization method seems to be an attractive method in this respect due to the fact that it not only retains the advantages of the conventional stochastic linearization method, such as simplicity and straightforward manner of derivation, but also may greatly improve the accuracy. Several authors recently studied the various versions of the improved stochastic linearization technique (Elishakoff and Zhang, 1991; Elishakoff, 1991; Zhang *et al.*, 1990; Fang and Fang, 1991). In this study a new stochastic linearization technique, as discussed in several references (Elishakoff and Zhang, 1991; Elishakoff, 1991; Zhang *et al.*, 1990), is extended to treat random vibrations of the nonlinearly deformed beam. The main idea of the new method consists of the requirement that the mean square value of the difference of potential energies of deformation, associated with the original nonlinear equation and its equivalent linear counterpart, should be minimal. It is instructive to first elucidate the basic idea on the example of the single degree-of-freedom system, governed by the following differential equation:

$$m\ddot{x}+c\dot{x}+g(x) = F(t) \tag{1}$$

where $F(t)$ is a random excitation resulting in stochastic response $x(t)$; $g(x)$ is a nonlinear deterministic function of displacement x. Within the stochastic linearization scheme, this differential equation is replaced by the "equivalent" linear equation

$$m\ddot{x}+c\dot{x}+k_{eq}x = F(t) \tag{2}$$

where the coefficient k_{eq} is determined through some suitable criterion of equivalence. In the linearization scheme utilized by Elishakoff and Zhang (1991), the equivalence criterion is chosen as follows:

$$E\{[U(x)-\tfrac{1}{2}k_{eq}x^2]^2\} = \min \tag{3}$$

where $U(x)$ is the potential energy of deformation of the original nonlinear structure, i.e.

$$U(x) = \int_0^x g(\alpha)\,\mathrm{d}\alpha. \tag{4}$$

This is accomplished by requiring

$$\frac{\mathrm{d}}{\mathrm{d}k_{eq}}E\{[U(x)-\tfrac{1}{2}k_{eq}x^2]^2\} = 0. \tag{5}$$

Equation (5) results in the following expression for the equivalent spring stiffness:

$$k_{eq} = \frac{2E[x^2U(x)]}{E(x^4)}. \tag{6}$$

In recent studies (Elishakoff and Zhang, 1991; Elishakoff, 1991; Zhang *et al.*, 1990), the authors have demonstrated the accuracy of this linearization technique by comparing the computed mean square displacements from different stochastic linearization methods with some known exact solutions. In the present study, the authors extend the above technique

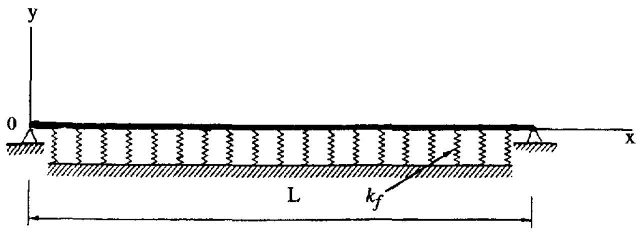

Fig. 1. Nonlinear beam on elastic foundation.

to continuous structures. The main idea is the same as the one described for the single degree-of-freedom system, except that the original continuous nonlinear structure is replaced by a multi degree-of-freedom linear system, and a set of equivalent spring stiffnesses are expressed by equations analogous to eqn (6) with x now corresponding to different modal displacements. The procedure will be elucidated in detail for random vibrations of the nonlinearly deformed beam.

In this paper we consider beams simply supported or clamped at their ends. Two loading conditions are considered: (a) the space-wise and time-wise white noise, in which case the exact solution is also obtained, (b) the space-wise uniformly distributed load and time-wise white noise, in which case no exact solution is available and the Monte Carlo simulations should be performed. For all cases and a wide variety of levels of excitation, the proposed method turns out to be superior to the classical stochastic linearization method.

2. FORMULATION OF THE PROBLEM

Consider a beam on an elastic foundation with pin-ended supports that are restrained from axial motion (Fig. 1). The beam is under a loading $q(x, t)$ which is space-wise and time-wise white noise with the following auto-correlation function:

$$R_{qq}(x_1, t_1; x_2, t_2) = 2\pi S_0 \delta(x_2 - x_1)\delta(t_2 - t_1). \tag{7}$$

The deflection is represented by the Fourier series in terms of mode shapes of the undamped beam

$$w(x, t) = \sum_{n=1}^{N} w_n(t) \sin\frac{n\pi x}{L}; \tag{8}$$

$w_n(t)$ is the modal contribution corresponding to nth mode. It is assumed that only the first N modes of the beam significantly contribute to formulating the response. However, it should be borne in mind that the assumption that the power spectral density of the load is that of white noise implies that all the modes are excited and contribute to the response of the beam; N is determined by the required accuracy in the evaluation of the specific response characteristic, such as mean square displacement or mean square stress. Crandall and Yildiz (1962) have shown that for the linearly deformed beam under white noise excitation, if the infinite series representing quantities such as mean square displacement, mean square stress, etc., converge, then the results can be made as accurate as desired by taking sufficiently large N. It is reasonable to expect similar results for the nonlinear problem. We consider a Bernoulli–Euler beam with transverse damping. Due to the fact that the equivalence criterion will utilize the concept of energy, we first formulate appropriate energies of the beam. Kinetic energy of the beam is given by, with eqn (8) taken into account,

$$T = \frac{\rho A}{2}\int_0^L \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x = \frac{\rho A L}{4}\sum_{n=1}^{N} (\dot{w}_n)^2\,; \tag{9}$$

the potential energy of bending deformation reads

$$V_{\mathrm{b}} = \frac{EI}{2}\int_0^L \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x = \frac{\pi^4 EI}{4L^3}\sum_{n=1}^{N} n^4 w_n^2. \tag{10}$$

The potential energy of stretching is given by

$$V_{\mathrm{s}} = \frac{AE}{2L}\left[\frac{1}{2}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\right]^2 = \frac{\pi^4 AE}{32L^3}\left[\sum_{n=1}^{N} n^2 w_n^2\right]^2 \tag{11}$$

whereas the potential energy of the deformation due to elastic foundation is

$$V_{\mathrm{e}} = \int_0^L \tfrac{1}{2}k_{\mathrm{f}} w^2\,\mathrm{d}x = \tfrac{1}{4}k_{\mathrm{f}} L \sum_{n=1}^{N} w_n^2 \tag{12}$$

where k_{f} is the translational stiffness of Winkler foundation. The potential function of a load is

$$V_{\mathrm{l}} = -\int_0^L q(x,t)w(x,t)\,\mathrm{d}x. \tag{13}$$

We expand the load in the series analogous to eqn (8)

$$q(x,t) = \sum_{n=1}^{N} q_n(t)\sin\frac{n\pi x}{L} \tag{14}$$

where again N terms have been retained. Then V_{l} becomes

$$V_{\mathrm{l}} = -\frac{L}{2}\sum_{n=1}^{N} q_n w_n. \tag{15}$$

The Lagrangian $\mathscr{L} = T - V$, where $V = V_{\mathrm{b}} + V_{\mathrm{s}} + V_{\mathrm{e}} + V_{\mathrm{l}}$, may now be written as

$$\mathscr{L} = \frac{\rho A L}{4}\sum_{n=1}^{N} (\dot{w}_n)^2 - \frac{\pi^4 EI}{4L^3}\sum_{n=1}^{N} n^4 w_n^2 - \frac{\pi^4 EA}{32L^3}\left[\sum_{n=1}^{N} n^2 w_n^2\right]^2 - \tfrac{1}{4}k_{\mathrm{f}} L\sum_{n=1}^{N} w_n^2 + \frac{L}{2}\sum_{n=1}^{N} q_n w_n. \tag{16}$$

The equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{w}_n}\right) - \frac{\partial \mathscr{L}}{\partial w_n} = 0 \quad (n = 1, 2, \ldots, N) \tag{17}$$

where w_n are considered as generalized coordinates. Substitution of eqn (16) into (17) yields the following N equations in terms of the modal amplitudes $w_n(t)$:

$$\ddot{w}_n+\frac{\beta}{\rho A}\dot{w}_n+\omega_0^2 n^4\left[1+\frac{\alpha}{n^4}+\frac{1}{4R^2n^2}\sum_{m=1}^{N} m^2 w_m^2\right]w_n = f_n \quad (n = 1,2,\ldots,N) \tag{18}$$

where β is an introduced linear viscous damping term. In addition, the following notations have been utilized:

$$\omega_0^2 = \frac{\pi^4 EI}{\rho A L^4}, \quad \alpha = \frac{k_f}{\rho A \omega_0^2}, \quad R = \sqrt{\frac{I}{A}}, \quad f_n = \frac{q_n}{\rho A}. \tag{19}$$

Equation (18) is a nonlinear stochastic differential equation. We seek the mean square response of the modal amplitudes. In this study, a new stochastic linearization technique described in the preceding section for the single degree-of-freedom system, is generalized to investigate the continuous structure at hand. The nonlinear system (18) is replaced by the following equivalent linear one:

$$\ddot{w}_n+\frac{\beta}{\rho A}\dot{w}_n+k_{eq}^{(n)}w_n = f_n(x) \quad (n = 1,2,\ldots,N). \tag{20}$$

In eqn (20), it is assumed that the replacing linear system is a decoupled one. The question of the decoupling will be addressed in detail in the Appendix.

In our problem, the total potential energy of the system (18) is

$$\begin{aligned} U(w_1,\ldots,w_N) &= \frac{\pi^4 EI}{2\rho A L^4}\sum_{n=1}^{N} n^4 w_n^2+\frac{k_f}{2\rho A}\sum_{n=1}^{N} w_n^2+\frac{\pi^4 E}{16\rho L^4}\left[\sum_{n=1}^{N} n^2 w_n^2\right]^2 \\ &= \frac{\omega_0^2}{2}\sum_{n=1}^{N} n^4 w_n^2+\frac{\alpha\omega_0^2}{2}\sum_{n=1}^{N} w_n^2+\frac{\omega_0^2}{16R^2}\left[\sum_{n=1}^{N} n^2 w_n^2\right]^2. \end{aligned} \tag{21}$$

We generalize the requirement of eqn (3), valid for the single degree-of-freedom system, for the continuous beam by requiring

$$E\left\{\left[U(w_1,w_2,\ldots,w_N)-\sum_{n=1}^{N}\tfrac{1}{2}k_{eq}^{(n)}w_n^2\right]^2\right\} = \min \tag{22}$$

which is achieved by using conditions

$$\frac{d}{dk_{eq}^{(m)}}\left\{E\left[U(w_1,w_2,\ldots,w_N)-\sum_{n=1}^{N}\tfrac{1}{2}k_{eq}^{(n)}w_n^2\right]^2\right\} = 0 \quad (m = 1,2,\ldots,N). \tag{23}$$

The conditions (23) are equivalent to

$$E\left\{\left[U(w_1,w_2,\ldots,w_N)-\sum_{n=1}^{N}\tfrac{1}{2}k_{eq}^{(n)}w_n^2\right]w_m^2\right\} = 0 \quad (m = 1,2,\ldots,N). \tag{24}$$

After some algebra, we arrive at the following expression for nonlinear spring constants:

$$\{k_{eq}\} = 2[A]^{-1}\{B\} \tag{25}$$

where

$$\{k_{eq}\}^T = \{k_{eq}^{(1)} \quad k_{eq}^{(2)} \quad \ldots \quad k_{eq}^{(N)}\}$$

$$\{B\}^T = \{E[w_1^2 U] \quad E[w_2^2 U] \quad \ldots \quad E[w_N^2 U]\}$$

$$[A] = \begin{bmatrix} E[w_1^2 w_1^2] & E[w_1^2 w_2^2] & \dots & E[w_1^2 w_N^2] \\ E[w_2^2 w_1^2] & E[w_2^2 w_2^2] & \dots & E[w_2^2 w_N^2] \\ \dots & \dots & \dots & \dots \\ E[w_N^2 w_1^2] & E[w_N^2 w_2^2] & \dots & E[w_N^2 w_N^2] \end{bmatrix}. \tag{26}$$

We denote

$$y_i = E[w_i^2]. \tag{27}$$

The Gaussian assumption for distribution of w_i and the subsequent conclusion that the equivalent system is decoupled (see the Appendix), leads to the independence between different modal amplitudes w_i and w_j $(i \neq j)$. Therefore

$$\begin{aligned} E[w_i^4] &= 3y^2 \\ E[w_i^2 w_j^2] &= y_i y_j \quad i \neq j. \end{aligned} \tag{28}$$

For simplicity, let us investigate the particular case $N = 3$. We have

$$[A]^{-1} = \frac{1}{10 y_1 y_2 y_3} \begin{bmatrix} \dfrac{4y_2 y_3}{y_1} & -y_3 & -y_2 \\ -y_3 & \dfrac{4y_1 y_3}{y_2} & -y_1 \\ -y_2 & -y_1 & \dfrac{4y_1 y_2}{y_3} \end{bmatrix}. \tag{29}$$

Consequently, we obtain

$$E[w_n^2 U] = \frac{\omega_0^2}{2} E\left[w_n^2 \sum_{m=1}^{3} m^4 w_m^2\right] + \frac{\alpha\omega_0^2}{2} E\left[w_n^2 \sum_{m=1}^{3} w_m^2\right] + \frac{\omega_0^2}{16R^2} E\left[w_n^2 \left(\sum_{m=1}^{3} m^2 w_m^2\right)^2\right]. \tag{30}$$

Under the assumption that the system is driven by zero mean Gaussian white noise q_n with spectral density S_0, i.e. f_n with spectral density $S_0/(\rho A)^2$, we obtain from eqn (20)

$$k_{\text{eq}}^{(n)} = \sigma_0^2 \omega_0^2 / y_n \tag{31}$$

or

$$\{k_{\text{eq}}\} = \sigma_0^2 \omega_0^2 \{1/y_1 \; 1/y_2 \dots 1/y_n\}^{\mathrm{T}} \tag{32}$$

where

$$\sigma_0^2 = \frac{L^4 S_0}{\pi^3 \beta E I}. \tag{33}$$

Substituting $k_{\text{eq}}^{(n)}$ in eqn (32) into eqn (25) and noting the fact that w_n is Gaussian yields

$$\begin{Bmatrix} \sigma_0^2 \\ \sigma_0^2 \\ \sigma_0^2 \end{Bmatrix} = \left(\frac{2}{\omega_0^2}\right)\left(\frac{1}{10y_1y_2y_3}\right)\begin{bmatrix} 4y_2y_3 & -y_1y_3 & -y_1y_2 \\ -y_2y_3 & 4y_1y_3 & -y_1y_2 \\ -y_2y_3 & -y_1y_3 & 4y_1y_2 \end{bmatrix}\begin{Bmatrix} E[w_1^2U] \\ E[w_2^2U] \\ E[w_3^2U] \end{Bmatrix} \tag{34}$$

where

$$\begin{aligned} E[w_1^2U] &= \frac{\omega_0^2}{2}y_1\,\{(3y_1+16y_2+81y_3)+\alpha(3y_1+y_2+y_3) \\ &\quad +\frac{1}{8R^2}(15y_1^2+48y_2^2+243y_3^2+24y_1y_2+72y_2y_3+54y_1y_3)\} \\ E[w_2^2U] &= \frac{\omega_0^2}{2}y_2\{(y_1+48y_2+81y_3)+\alpha(y_1+3y_2+y_3) \\ &\quad +\frac{1}{8R^2}(3y_1^2+240y_2^2+243y_3^2+24y_1y_2+216y_2y_3+18y_1y_3)\} \\ E[w_3^2U] &= \frac{\omega_0^2}{2}y_3\,\{(y_1+16y_2+243y_3)+\alpha(y_1+y_2+3y_3) \\ &\quad +\frac{1}{8R^2}(3y_1^2+48y_2^2+1215y_3^2+8y_1y_2+216y_2y_3+54y_1y_3)\}. \end{aligned} \tag{35}$$

Substitution of eqn (35) into eqn (34) results in a set of algebraic nonlinear equations for y_1, y_2 and y_3. For different excitation levels characterized by σ_0, the foundation modulus α and various radii of gyration R, eqn (34) can be solved numerically. In our study, the standard Levenberg–Marquardt algorithm is implemented.

With $y_i = E[w_i^2]$ obtained, we arrive at the mean square response of the beam as

$$E[w^2(x,t)] = \sum_{n=1}^{3}\sum_{m=1}^{3} E(w_m w_n)\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}. \tag{36}$$

However, the modal amplitudes w_m are uncorrelated. Therefore, while the membrane stress causes the modal amplitudes to become statistically dependent, it still leaves them uncorrelated. Equation (36) then reduces to

$$E[w^2\,(x,t)] = \sum_{m=1}^{3} y_m \sin^2\frac{m\pi x}{L}. \tag{37}$$

3. COMPARISON WITH OTHER METHODS

3.1. *Fokker–Planck equation method*

From the solution of the Fokker–Planck equation, the exact expression of the probability density function is obtained as

$$p\,(w_1,w_2,\ldots,w_N) = \frac{1}{c}\exp\left\{-\frac{1}{2\sigma_0^2}\left[\sum_{n=1}^{N} n^4w_n^2 + \frac{1}{8R^2}\sum_{n=1}^{N}\sum_{m=1}^{N} m^2n^2w_m^2w_n^2 + \alpha\sum_{n=1}^{N} w_n^2\right]\right\} \tag{38}$$

where c is the normalization factor, i.e.

$$c = \int_{-\infty}^{\infty} \mathrm{d}w_1 \ldots \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma_0^2}\left[\sum_{n=1}^{N} n^4 w_n^2 + \frac{1}{8R^2}\sum_{n=1}^{N}\sum_{m=1}^{N} m^2 n^2 w_m^2 w_n^2 + \alpha \sum_{n=1}^{N} w_n^2\right]\right\} \mathrm{d}w_N. \tag{39}$$

Equation (38) coincides with that of Herbert (1964) except that we have introduced an additional term associated with the elastic foundation. Hence, the modal mean square responses are obtained by integration

$$E[w_m^2] = \frac{1}{c}\int_{-\infty}^{\infty} \mathrm{d}w_1 \ldots \int_{-\infty}^{\infty} w_m^2 \exp\left\{-\frac{1}{2\sigma_0^2}\left[\sum_{n=1}^{N} n^4 w_n^2 + \frac{1}{8R^2}\sum_{r=1}^{N}\sum_{n=1}^{N} n^2 r^2 w_n^2 w_r^2 + \alpha \sum_{n=1}^{N} w_n^2\right]\right\} \mathrm{d}w_N. \tag{40}$$

Equation (40) must be evaluated by numerical integration.

3.2. *Conventional stochastic linearization technique*

By the conventional equivalent linearization method, we obtain

$$k_{\mathrm{eq}}^{(n)} = n^4\omega_0^2 + \alpha\omega_0^2 + \frac{\omega_0^2 n^2}{4R^2}\,\frac{E\left[w_n^2 \sum_{m=1}^{N} m^2 w_m^2\right]}{E[w_n^2]}. \tag{41}$$

From the equivalent linear system (20), we have eqn (31). Substituting $k_{\mathrm{eq}}^{(n)}$ in eqn (41) into eqn (31) yields

$$\sigma_0^2 = (n^4+\alpha)E[w_n^2] + \frac{n^2}{4R^2}E\left[w_n^2 \sum_{m=1}^{N} m^2 w_m^2\right] (n = 1, \ldots, N). \tag{42}$$

If only the first three modes are considered, in view of eqn (27), we obtain

$$\sigma_0^2 = (1+\alpha)y_1 + \frac{1}{4R^2}(3y_1^2 + 4y_1y_2 + 9y_1y_3)$$

$$\sigma_0^2 = (16+\alpha)y_2 + \frac{1}{R^2}(y_1y_2 + 12y_2^2 + 9y_2y_3)$$

$$\sigma_0^2 = (81+\alpha)y_3 + \frac{9}{4R^2}(y_1y_3 + 4y_2y_3 + 27y_3^2). \tag{43}$$

For specific values of σ_0^2, α and R, one can evaluate $E[w_1^2]$, $E[w_2^2]$ and $E[w_3^2]$ through solving the set of nonlinear equations eqn (43).

4. RESULTS AND DISCUSSION

Numerical computations have been performed for the mean square deflection at the midspan of the beam for various values of the three parameters σ_0^2, α and R. The results from the three methods are presented in Figs 2–5. As pointed out by Seide (1975, 1976, 1988), Seide and Aokimi (1983), Seide and Tehranizadeh (1988) and as is confirmed in eqn (37), accurate values of the mean square deflections can be obtained by using a three term approximation, while the number of terms required for accurate stress values may be much larger. In the present study, only the first three modal displacements are considered, with the emphasis on demonstrating the effectiveness of the new technique.

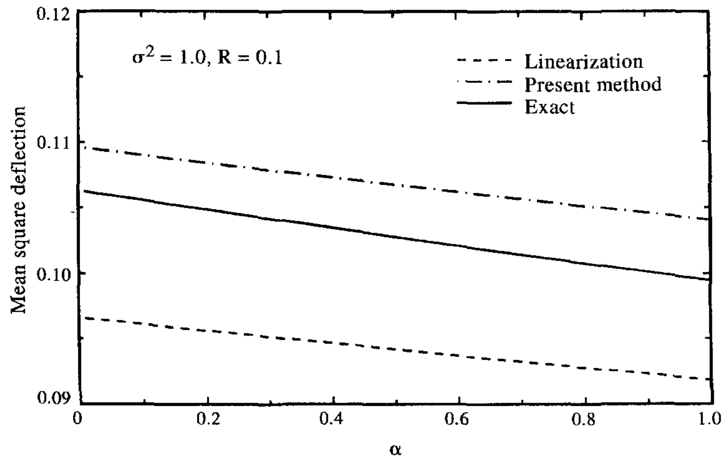

Fig. 2. Effect of foundation stiffness (α) on the mean square deflection at the midspan of the simply supported beam (space-wise white noise loading).

As is seen from the differential equation (18), when R tends to infinity the effect of the nonlinearity disappears. Therefore, the magnitude of $1/R$ can be viewed as the parameter related to the magnitude of nonlinearity. The effect of the foundation stiffness on the mean square maximum deflection at the midspan of the beam is shown in Fig. 2 for one set of levels of excitation and nonlinearity. It can be seen that the new equivalent linearization method yields more accurate results than the conventional technique. Also, it is shown that the new method usually gives greater values of mean square deflection than the exact solution, while the conventional method yields values below the exact one. Furthermore, when the stiffness of the foundation k_f becomes larger (and consequently the parameter α is larger too), both methods tend to produce equally accurate results. This conclusion should have been anticipated in view of the fact that the system is "more" linear in this case, due to linearity of the Winkler foundation model. However, when α is small, the new stochastic linearization method achieves a much better estimate of the mean square deflection than the conventional technique. The mean square deflections vs parameter R are shown in Figs 3 and 4; the new method performs much better than the conventional stochastic linearization technique for the relatively high nonlinearity of the system, i.e. R is from 0 to 1.

The superior performance of the energy based linearization can be partially explained by the fact that the exact probability density given in eqn (38) is expressible in terms of the strain energy of the system, given in eqn (21). Indeed, eqn (38) can be rewritten as follows:

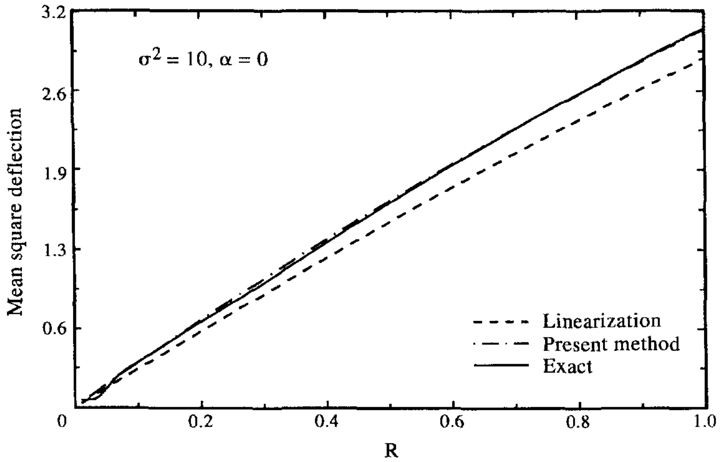

Fig. 3. Effect of nonlinearity (R) on the mean square deflection at the midspan of the simply supported beam ($\sigma_0^2 = 10, \alpha = 0$) (space-wise white noise loading).

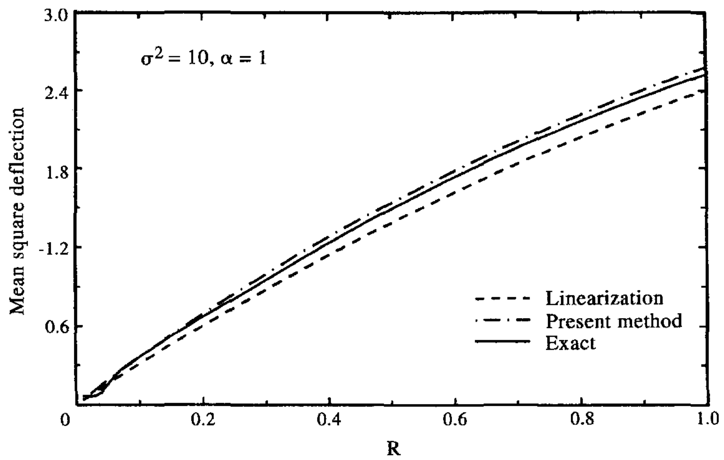

Fig. 4. Effect of nonlinearity (R) on the mean square deflection at the midspan of the simply supported beam ($\sigma_0^2 = 10, \alpha = 1$) (space-wise white noise loading).

$$p(w_1, w_2, \cdots, w_N) = \frac{1}{c}\exp\left[-\frac{1}{\sigma_0^2\omega_0^2}U(w_1^2, \ldots, w_N^2)\right] \tag{44}$$

where the expression of strain energy is given in eqn (21). Note that this interesting feature for the single degree-of-freedom systems was discussed by several authors. One may consult, for example, Caughey (1963) and eqn (8.35.a) in Lin (1967).

The effect of the strength of excitation on the accuracy of the two methods is shown in Fig. 5, where the vertical axis denotes the percentage error of the mean square deflection at the midspan of the beam either between the conventional linearization method or the new linearization method, and the exact solution; the new approach achieves more accurate results than the conventional technique for all the excitation levels.

To get additional insight into the performance of the proposed method, the other set of boundary conditions was investigated, namely, the beam clamped at both ends under both the space-wise and time-wise white noise. Figure 6 portrays the mean square displacement calculated by the proposed stochastic linearization method, conventional stochastic linearization method and exact solution. An exact solution follows the derivation

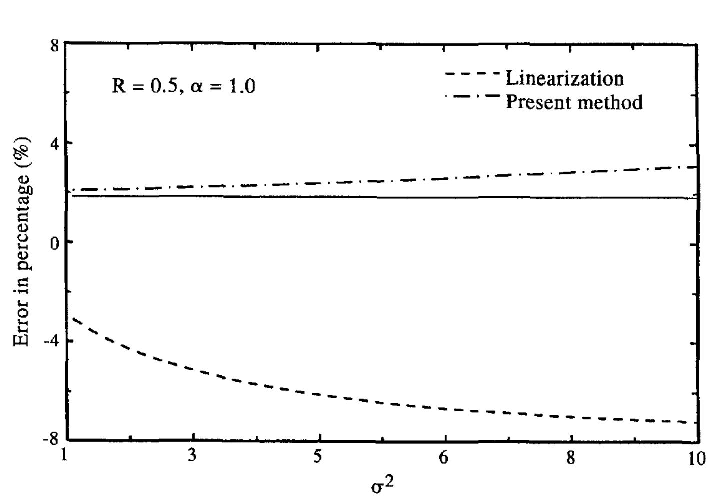

Fig. 5. Percentage error in the determination of the mean square deflection by the two methods in comparison with the exact solution (space-wise white noise loading).

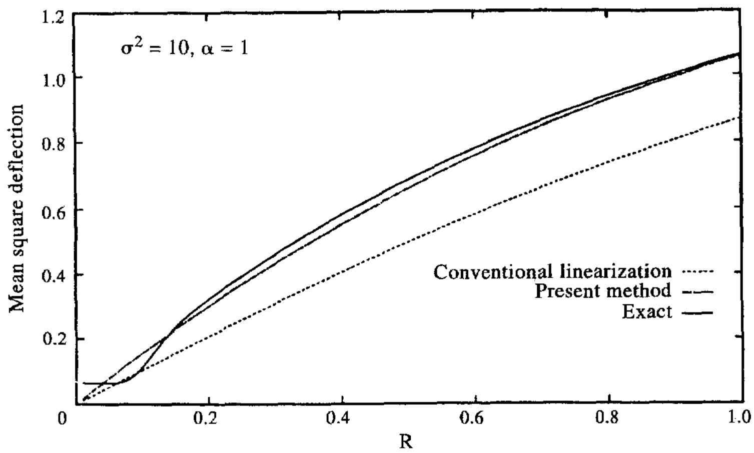

Fig. 6. Effect of nonlinearity (R) on the mean square deflection at the midspan of the beam clamped at both ends ($\sigma_0^2 = 10, \alpha = 1$) (space-wise white noise loading).

given in eqns (14)–(32) except that instead of the sinusoidal mode shape in eqn (8), the following mode shape is utilized:

$$\psi_j(x) = \cosh \gamma_j x - \cos \gamma_j x - \alpha_j(\sinh \gamma_j x - \sin \gamma_j x) \tag{45}$$

where

$$\alpha_j = \begin{cases} \tanh \frac{1}{2}\gamma_j & \text{if } m \text{ is odd} \\ \coth \frac{1}{2}\gamma_j & \text{if } m \text{ is even} \end{cases} \tag{46}$$

and the values of γ_j are the consecutive solutions of the transcendental equation

$$\cosh \gamma_j \cos \gamma_j = 1. \tag{47}$$

As is seen from Fig. 6 for the clamped beam too, the proposed method results in the mean square response which is much closer to the exact solution than the classical linearization method.

In both cases of the beams considered, the exact solutions were also obtained. The natural question arises: how does the proposed method perform when the exact solution is not available? To answer this question, the additional loading condition was also investigated, namely, the load $q(x, t)$ was represented as a product $r(x)q(t)$. Whereas $q(t)$ was assumed to be a weakly stationary Gaussian random process namely white noise, $r(x)$ was taken as a deterministic function. Specifically $r(x)$ was taken as a constant, representing space-wise uniformly distributed load. This representation is valid for members of relatively short length when the correlation length of the excitation is much greater than the length of the beam. For such a loading condition, an exact solution is unavailable and instead Monte Carlo simulations should be conducted to check the accuracy of the proposed stochastic linearization. Figure 7 depicts results of such a comparison for the simply supported beam. As is clearly seen, the proposed method again exhibits much higher accuracy than the conventional linearization technique.

To sum up, for different boundary conditions and the loading patterns, the suggested method is superior to the classical stochastic linearization technique, especially in the important high nonlinearity range of the parameters.

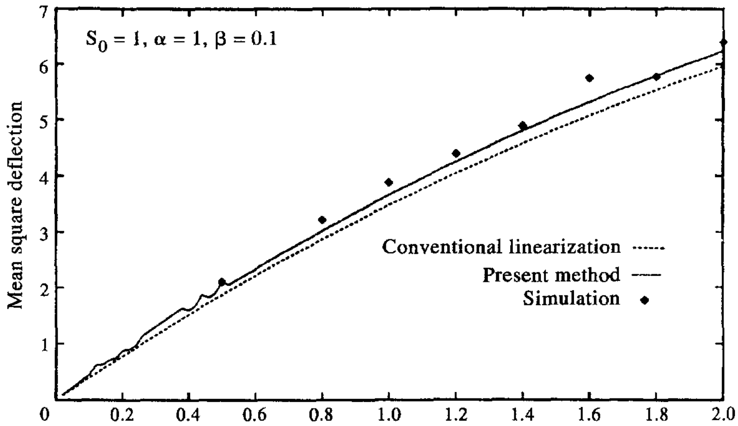

Fig. 7. The variation of the mean square deflection at the midspan of the simply supported beam with the nonlinearity coefficient R (space-wise uniformly distributed load).

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APPENDIX

Uncoupledness of equivalent linear system

Under the assumption that the modal displacements are normally distributed, one can show that the equivalent linear system is uncoupled. Indeed, suppose that the equivalent linear system is governed by the vector equation

$$M\ddot{w}+C\dot{w}+Kw=f \tag{A1}$$

where M and C are diagonal mass and damping matrices, K is a non-diagonal stiffness matrix, and fs are independent white noises. New equivalence criterion requires

$$E\{[U(w)-\tfrac{1}{2}w^{\mathrm{T}}Kw]^2\} = \min \tag{A2}$$

where

$$w = \{w_1 \quad w_2 \quad \cdots \quad w_N\}^{\mathrm{T}}$$
$$K = [k_{ij}]. \tag{A3}$$

The condition that the derivatives of (A2) with respect to k_{ij} equal zero leads to

$$E[ww^{\mathrm{T}}\,K\,ww^{\mathrm{T}}] = 2E[wU(w)w^{\mathrm{T}}]. \tag{A4}$$

For simplicity, let us consider the two degree-of-freedom system. Suppose

$$K=\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22}\end{bmatrix}. \tag{A5}$$

Then

$$ww^{\mathrm{T}}\,K\,ww^{\mathrm{T}} = \begin{bmatrix} w_1^4k_{11}+2w_1^3w_2k_{12}+w_1^2w_2^2k_{22} & w_1^3w_2k_{11}+2w_1^2w_2^2k_{12}+w_1w_2^3k_{22} \\ w_1^3w_2k_{11}+2w_1^2w_2^2k_{12}+w_1w_2^3k_{22} & w_1^2w_2^2k_{11}+2w_1w_2^3k_{12}+w_2^4k_{22}\end{bmatrix}$$

$$E[wU(w)w^{\mathrm{T}}] = \begin{bmatrix} w_1^2U & w_1w_2U \\ w_1w_2U & w_2^2U\end{bmatrix}. \tag{A6}$$

Therefore eqn (A4) becomes

$$[A_1]\{k_1\} = 2\{B_1\} \tag{A7}$$

where

$$[A_1] = \begin{bmatrix} E[w_1^4] & 2E[w_1^3w_2] & E[w_1^2w_2^2] \\ E[w_1^3w_2] & 2E[w_1^2w_2^2] & E[w_1w_2^3] \\ E[w_1^2w_2^2] & 2E[w_1w_2^3] & E[w_2^4]\end{bmatrix}$$

$$\{k_1\} = \{k_{11} \quad k_{12} \quad k_{22}\}^{\mathrm{T}}$$

$$\{B_1\} = \{E[w_1^2U] \quad E[w_1w_2U] \quad E[w_2^2U]\}^{\mathrm{T}}. \tag{A8}$$

The assumption of zero mean Gaussian distribution of the modes w_n implies

$$E[w_1^3w_2] = 0$$
$$E[w_1w_2^3] = 0. \tag{A9}$$

In view of eqn (21), we also have

$$E[w_1w_2U] = 0. \tag{A10}$$

As a result, A_1 and B_1 turn out to be

$$[A_1] = \begin{bmatrix} E[w_1^4] & 0 & E[w_1^2 w_2^2] \\ 0 & 2E[w_1^2 w_2^2] & 0 \\ 2E[w_1^2 w_2^2] & 0 & E[w_2^4] \end{bmatrix}$$

$$\{B_1\} = \{E[w_1^2 U] \quad 0 \quad E[w_2^2 U]\}^{\mathrm{T}}. \tag{A11}$$

Substituting $[A_1]$, $\{B_1\}$ in eqn (A11) into eqn (A7), one obtains

$$k_{12} = 0. \tag{A12}$$

This illustrates the uncoupledness of the equivalent linear system in the two degree-of-freedom setting. An analogous proof holds for $N > 2$.